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Some nonunitary, indecomposable representations of the Euclidean algebra $\mathfrak{e}(3)$

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Abstract

The Euclidean group $E(3)$ is the noncompact, semidirect product group $E(3) \cong \mathbb{R}^3 \rtimes SO(3)$. It is the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The Euclidean algebra $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. We construct three distinct families of finite-dimensional, nonunitary representations of $\mathfrak{e}(3)$ and show that each representation is indecomposable. The representations of the first family are explicitly realized as subspaces of the polynomial ring $\mathbb{F}[X, Y, Z]$ with the action of $\mathfrak{e}(3)$ given by differential operators. The other families are constructed via duals and tensor products of the representations within the first family. We describe subrepresentations, quotients and duals of these indecomposable representations.

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1. Introduction

The Euclidean group $E(3)$ is the noncompact, semidirect product group $E(3) \cong \mathbb{R}^3 \rtimes SO(3)$. It is the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The history of this group is intertwined with the early history of physics and Euclidean geometry and dates back well before the concept of a group was even invented. Some of the more remarkable geometrical features associated with subgroups of $E(3)$, e.g. the existence of Platonic solids, have been the subject of human fascination since antiquity.

The representations of $E(3)$ play an important role in the representation theory of the Poincaré group [1]. $E(3)$ appears in the study of crystallographic groups of physics, which are subgroups of $E(3)$. The group has also been studied outside of physics or mathematics: it is used for instance in the mathematical description of robotic manipulations [2]. A lesser known application is due to Fock [3], who showed that $E(3)$ is the symmetry group of the zero-energy solutions of the Schrödinger equation for the hydrogen atom.

The Euclidean algebra $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. Although irreducible representations of $\mathfrak{e}(3)$ have been studied extensively (see for example [4–7]), little is known about its finite-dimensional, indecomposable representations.

We remind the reader that a representation V is *irreducible* if it has no subrepresentations other than $\{0\}$, and V. It is *completely reducible* if it is isomorphic to a direct sum of irreducible representations. It is *indecomposable* if it is not isomorphic to a direct sum of two nonzero subrepresentations ([8], p 5).

In this paper, we construct three infinite families of finite-dimensional indecomposable $\mathfrak{e}(3)$ representations. The representations of the first family are explicitly realized as subspaces of the polynomial ring $\mathbb{F}[X, Y, Z]$ with the action of $\mathfrak{e}(3)$ given by differential operators. The other families are constructed via duals and tensor products of the representations within the first family. All nontrivial examples in each family are nonunitary.

The organization of the paper is as follows. In section 2, we describe the Lie group $E(3)$ and the complexification of its Lie algebra $\mathfrak{e}(3)$. Section 3 records results about $\mathfrak{sl}(2, \mathbb{C})$, which is a subalgebra of $\mathfrak{e}(3)$, and its representations that will be used in following sections. In section 4, we construct three families of representations of $\mathfrak{e}(3)$ and prove they are indecomposable. We also describe the subrepresentations, quotients and duals of these representations.

2. The Euclidean group $E(3)$ and the Euclidean algebra $\mathfrak{e}(3)$

The Euclidean group $E(3)$ may be realized as a subgroup of $GL(4, \mathbb{R})$:

$$E(3) \cong \left\{ \begin{pmatrix} & & & x_1 \\ & R & & x_2 \\ & & & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} | R \in SO(3), \text{ and } x_1, x_2, x_3 \in \mathbb{R} \right\}, \tag{1}$$

where $SO(3)$ is the special orthogonal group of 3×3 real matrices X with the determinant one such that $X^{\text{tr}} = X^{-1}$ (X^{tr} is the transpose of X). It can then be shown (see [9]) that the Lie algebra of $E(3)$ is the space of all 4×4 real matrices of the form

$$\begin{pmatrix} & & & x_1 \\ & R & & x_2 \\ & & & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

where $R \in \mathfrak{so}(3)$ and $x_1, x_2, x_3 \in \mathbb{R}$: $\mathfrak{so}(3)$ is the special orthogonal algebra consisting of 3×3 real matrices X such that $X^{\text{tr}} = -X$.

The *Euclidean algebra* $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. It may be described abstractly with basis $J_z, J_+, J_-, P_z, P_+, P_-$, and commutation relations

$$[J_z, J_\pm] = \pm 2J_\pm, \qquad [J_+, J_-] = J_z \tag{3}$$
$$[J_z, P_\pm] = \pm 2P_\pm, \tag{4}$$
$$[J_\pm, P_z] = -P_\pm, \qquad [J_z, P_z] = 0 \tag{5}$$
$$\left[P_i, P_j\right] = 0, \tag{6}$$
$$[J_+, P_+] = [J_-, P_-] = 0, \quad [J_-, P_+] = [J_+, P_-] = -2P_z. \tag{7}$$

Note that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3)_{\mathbb{C}}$, where $\mathfrak{so}(3)_{\mathbb{C}}$ is the complexification of $\mathfrak{so}(3)$. Further, $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{e}(2)$ are subalgebras of $\mathfrak{e}(3)$. Specifically, $\mathfrak{sl}(2, \mathbb{C}) \cong \langle J_z, J_+, J_- \rangle$ and $\mathfrak{e}(2) \cong \langle J_z, P_+, P_- \rangle$.

3. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and its irreducible representations

The special linear algebra $\mathfrak{sl}(2, \mathbb{C})$ is the Lie algebra of traceless 2×2 matrices with complex entries, and Lie bracket given by the commutator. It may be described abstractly with basis J_z, J_+, J_- and commutation relations

$$[J_z, J_+] = 2J_+, \qquad [J_z, J_-] = -2J_-, \qquad [J_+, J_-] = J_z. \tag{8}$$

The finite-dimensional, irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules play a central role in our construction of finite-dimensional, indecomposable representation of $\mathfrak{e}(3)$ below. Thus, for completeness and to establish notation, we briefly review the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

Following ([10], exercise 7.4), the finite-dimensional, irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules may be realized as subspaces of the polynomial ring $\mathbb{F}[X, Y]$, and the $\mathfrak{sl}(2, \mathbb{C})$-action given by differential operators. Specifically, for each natural number n, there is an irreducible $(n + 1)$-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module denoted by $M_n[X, Y]$ with basis,

$$\{X^n, X^{n-1}Y, X^{n-2}Y^2, \ldots, XY^{n-1}, Y^n\}, \tag{9}$$

consisting of the $n + 1$ monomials of degree n. The action of $\mathfrak{sl}(2, \mathbb{C})$ on $M_n[X, Y]$ is given by the differential operators:

$$J_z = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}, \quad J_+ = X\frac{\partial}{\partial Y}, \quad J_- = Y\frac{\partial}{\partial X}. \tag{10}$$

The category of finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules is semisimple; hence if V and W are finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules, $V \otimes W$ is completely reducible. The tensor product decomposition of irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules is given by the Clebsch–Gordan theorem ([9], theorem D.1).

Theorem 1. *Let $n \leqslant m$; then*

$$M_m[X, Y] \otimes M_n[X, Y] \cong \bigoplus_{i=0}^{n} M_{m+n-2i}[X, Y]. \tag{11}$$

Remark 1. Since $\mathfrak{sl}(2, \mathbb{C})$ is a subalgebra of $\mathfrak{e}(3)$, if V is a finite-dimensional $\mathfrak{e}(3)$-module, then V, when restricted to $\mathfrak{sl}(2, \mathbb{C})$, decomposes into irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules $V \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_m M_m[X, Y]$.

4. Finite-dimensional, indecomposable $\mathfrak{e}(3)$-modules

We now construct three families of finite-dimensional representations of $\mathfrak{e}(3)$. For each $n, m \in \mathbb{Z}_{\geqslant 0}$ with $n \leqslant m$, we construct two representations: a *raising string* denoted by $R[n, m]$ and a *lowering string* $L[n, m]$. The raising string representations are constructed explicitly as subspaces of the polynomial ring $\mathbb{F}[X, Y, Z]$ with the action of $\mathfrak{e}(3)$ given by differential operators. The lowering string representations are the duals of the raising string representations. The third distinct family of indecomposable representations is created by taking the tensor product of a raising with a lowering string. We call members of the third family *parallelogram representations*. We show that each family consists of indecomposable representations and consider subrepresentations, quotients and duals. These families generalize to $\mathfrak{e}(3)$ the similar ideas presented for $\mathfrak{e}(2)$ in [11], and [12].

4.1. Raising string representations

Fix natural numbers n and m with $n \leqslant m$, and consider the $(n+1)(m+1)$-dimensional subspace of the polynomial ring $\mathbb{F}[X, Y, Z]$:

$$R[n, m] \equiv \bigoplus_{j=0}^{n} M_{m-n+2j}[X, Y]Z^{n-j}, \tag{12}$$

where $M_{m-n+2j}[X, Y]Z^{n-j} = \{p(X, Y)Z^{n-j} : p(X, Y) \in M_{m-n+2j}[X, Y]\}$. We may define an action of $\mathfrak{e}(3)$ in terms of differential operators on $R[n, m]$; in addition to the explicit action of equation (10), we have

$$P_z = -XY\frac{\partial}{\partial Z}, \qquad P_+ = X^2\frac{\partial}{\partial Z}, \qquad P_- = Y^2\frac{\partial}{\partial Z}. \tag{13}$$

Observe that

$$P \cdot M_{m-n+2j}[X, Y]Z^{n-j} \subseteq M_{m-n+2j+2}[X, Y]Z^{n-j-1} \tag{14}$$

for $0 \leqslant j \leqslant n-1$ and

$$P \cdot M_{m+n}[X, Y] = 0, \qquad \text{where } P \in \{P_z, P_+, P_-\}. \tag{15}$$

In a straightforward manner one can now check that the commutations relations of $\mathfrak{e}(3)$ are respected by the action described in equations (10) and (13) on the vector space $R[n, m]$ given in equation (12), so that we have defined an $\mathfrak{e}(3)$-representation. Indeed, P decreases the degree in Z but increases the degree in X and Y and so raises the dimensionality of the $\mathfrak{sl}(2, \mathbb{C})$-module. Thus,

Definition 1. *For all $n, m \in \mathbb{Z}_{\geqslant 0}$, where $n \leqslant m$, the $(n+1)(m+1)$-dimensional $\mathfrak{e}(3)$-representation $R[n, m]$ described in equations (12), (10) and (13) is a raising string representation.*

Proposition 1. *$R[n, m]$ is indecomposable.*

Proof. To establish indecomposability, it suffices to show that any nonzero element of $R[n, m]$ generates the subrepresentation $M_{m+n}[X, Y]$.

Let $x = \sum_{j=0}^{n} \alpha_j v_j$, where α_j is a scalar, $v_j \in M_{m-n+2j}[X, Y]Z^{n-j}$ and $v_j \neq 0$ be an arbitrary nonzero element of $R[n, m]$. Let l be minimal among $\{0, 1, 2, \ldots, n\}$ such that $\alpha_l \neq 0$. Then, $(P_z)^{n-l} \cdot x$ is a nonzero element of $M_{m+n}[X, Y]$. Since $M_{m+n}[X, Y]$ is an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module, any nonzero element of $M_{m+n}[X, Y]$ generates all of $M_{m+n}[X, Y]$. □

Proposition 2. *V is a subrepresentation of $R[n, m]$ if and only if $V \cong R[n-t, m+t]$, such that $t \leqslant n$.*

Proof. Suppose $V \cong R[n-t, m+t]$ such that $0 \leqslant t \leqslant n$. Then, since

$$R[n, m] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{n} M_{m-n+2j}[X, Y]Z^{n-j} \tag{16}$$

$$\cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{t-1} M_{m-n+2j}[X, Y]Z^{n-j} \oplus M_{m-n+2t}[X, Y]Z^{n-t} \oplus \cdots \oplus M_{m+n}[X, Y] \tag{17}$$

$$\cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{t-1} M_{m-n+2j}[X, Y]Z^{n-j} \oplus R[n-t, m+t], \tag{18}$$

and $P \cdot M_{m-n+2j}[X, Y]Z^{n-j} \subseteq M_{m-n+2j+2}[X, Y]Z^{n-j-1}$ for $P \in \{P_z, P_-, P_+\}$, it is clear that $R[n-t, m+t]$ is a subrepresentation of $R[n, m]$.

Conversely, suppose that V is a subrepresentation of $R[n, m]$. Let $\{b_i\}_i$ be a basis of V. Each b_i may be written $b_i = \sum_{j=0}^{n} \alpha_{i,j} v_{i,j}$, where $v_{i,j} \in M_{m-n+2j}[X, Y]Z^{n-j}$, $v_{i,j} \neq 0$ and the $\alpha_{i,j}$ are scalars. There is a minimal $t \leqslant n$ such that $\alpha_{i,t} \neq 0$ for some i (perhaps more than one i). Then $V \subseteq R[n-t, m+t]$, considering $P \cdot M_{m-n+2j}[X, Y]Z^{n-j} \subseteq M_{m-n+2j+2}[X, Y]Z^{n-j-1}$.

Let b_s be a basis vector such that $\alpha_{s,t} \neq 0$. Then we may write $b_s = \sum_{j=t}^{n} \alpha_{s,t} v_{s,j}$. Then $(P_z)^N \cdot b_s$ has a nonzero component of $M_{m-n+2(t+N)}[X, Y]Z^{n-(t+N)}$, where $N = 0, \ldots, n-t$, and does not have a component of $M_k[X, Y]Z^{m-k+j}$ for $k < m-n+2(t+N)$. In particular, $(P_z)^{n-t} \cdot b_s$ is a nonzero element of $M_{m+n}[X, Y]$, and hence generates all of $M_{m+n}[X, Y]$. Then, a linear combination of elements in $M_{m+n}[X, Y]$ together with $(P_z)^{n-t-1} \cdot b_s$ generate a nonzero element of $M_{m+n-2}[X, Y]Z$, and hence all of $M_{m+n-2}[X, Y]Z$. Continuing in this manner we generate all of $R[n-t, m+t]$, and thus $V \cong R[n-t, m+t]$. □

Proposition 2 implies the following proposition:

Proposition 3. *$R[n, m]$ is irreducible if and only if $n = 0$.*

For $1 \leqslant t \leqslant n$, consider the $\mathfrak{sl}(2, \mathbb{C})$-decomposition of $R[t-1, m-n+t-1]$, $R[n, m]$ and $R[n-t, m+t]$:

$$R[t-1, m-n+t-1] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{t-1} M_{m-n+2j}[X, Y]Z^{n-j}, \tag{19}$$

$$R[n, m] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{t-1} M_{m-n+2j}[X, Y]Z^{n-j} \oplus \bigoplus_{j=t}^{n} M_{m-n+2j}[X, Y]Z^{n-j}, \tag{20}$$

$$R[n-t, m+t] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=t}^{n} M_{m-n+2j}[X, Y]Z^{n-j}. \tag{21}$$

From this it is clear that the canonical embedding of $R[t-1, m-n+t-1]$ into $\frac{R[n,m]}{R[n-t,m+t]}$ is one-to-one, and onto (and commutes with the action of $\mathfrak{e}(3)$), giving us the next proposition.

Proposition 4. *Let $R[n-t, m+t]$ for $1 \leqslant t \leqslant n$ be a subrepresentation of $R[n, m]$; then,*

$$\frac{R[n, m]}{R[n-t, m+t]} \cong R[t-1, m-n+t-1]. \tag{22}$$

The final result of the subsection follows from a consideration of the $\mathfrak{sl}(2, \mathbb{C})$-decomposition of $R[n, m]$ and $R[k, l]$. The result implies that the family of raising string representations is indeed infinite.

Proposition 5. *$R[n, m] \cong R[k, l]$ if and only if $n = k$ and $m = l$.*

Figure 1 illustrates the action of P_+, P_z and P_- (up to scalar multiple) in $R[3, 3] \cong_{\mathfrak{sl}(2,\mathbb{C})} M_0[X, Y]Z^3 \oplus M_2[X, Y]Z^2 \oplus M_4[X, Y]Z \oplus M_6[X, Y]$. Vertices on the same vertical line form a basis of an $\mathfrak{sl}(2, \mathbb{C})$-irrep. Vertices of the same horizontal height have the same J_z-weight.

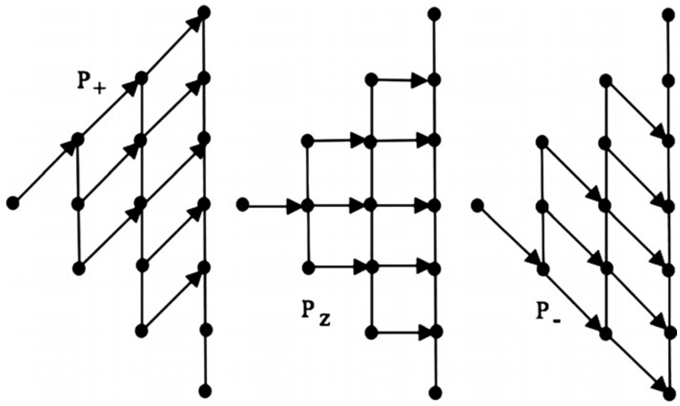

Figure 1. The action of P_+, P_z and P_- (up to scalar multiple) in $R[3, 3]$.

4.2. Lowering string representations

Let V be a finite-dimensional $\mathfrak{e}(3)$-module. Then the *dual* or *contragradient* vector space V^* is also an $\mathfrak{e}(3)$-module with action given by ([10], p 26)

$$(x \cdot f)(v) = -f(x \cdot v), \quad \text{where } f \in V^*, \quad v \in V \quad \text{and} \quad x \in \mathfrak{e}(3). \tag{23}$$

Definition 2. *For each $n, m \in \mathbb{Z}_{\geqslant 0}$ and $n \leqslant m$, a lowering string representation, denoted by $L[n, m]$, is the dual of the raising string representation $R[n, m]$. That is, $L[n, m] \equiv R[n, m]^*$.*

We will now describe the representation $L[n, m]$. Define a basis $\{f_{j,i}\}_{j=0,\ldots,n;i=0,\ldots,m-n+2j}$ of $L[n, m] = R[n, m]^*$, where $f_{j,i} \in R[n, m]^*$ is given by

$$f_{a,b}(X^{m-n+2j-i}Y^iZ^{n-j}) = \begin{cases} 1 & : a = j, b = i \\ 0 & : \text{otherwise.} \end{cases} \tag{24}$$

The action of $\mathfrak{e}(3)$ on the basis is given by

$$\begin{aligned} J_z \cdot f_{j,i} &= -(m - n + 2j - 2i) f_{j,i} \\ J_+ \cdot f_{j,i} &= -(i + 1) f_{j,i+1} \\ J_- \cdot f_{j,i} &= -(m - n + 2j - i + 1) f_{j,i-1} \end{aligned} \tag{25}$$

$$\begin{aligned} P_z \cdot f_{j,i} &= (n - j + 1) f_{j-1,i-1} \\ P_+ \cdot f_{j,i} &= -(n - j + 1) f_{j-1,i} \\ P_- \cdot f_{j,i} &= -(n - j + 1) f_{j-1,i-2}. \end{aligned} \tag{26}$$

Fixing j, $0 \leqslant j \leqslant n$, define $L_j[n, m] \equiv \langle f_{j,i}\rangle_{i=0,\ldots,m-n+2j}$. Then,

$$L_j[n, m] \cong_{\mathfrak{sl}(2,\mathbb{C})} M_{m-n+2j}[X, Y] \tag{27}$$

and

$$L[n, m] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{n} L_j[n, m]. \tag{28}$$

In addition,

$$P \cdot L_j[n, m] \subseteq L_{j-1}[n, m], \tag{29}$$

where $P \in \{P_z, P_+, P_-\}$ for $1 \leqslant j \leqslant n$ and

$$P \cdot L_0[n,m] = 0, \qquad \text{where } P \in \{P_z, P_+, P_-\}. \tag{30}$$

The proof of the indecomposability of $L[n,m]$ and proofs for the results involving subrepresentations and quotients of lowering strings follow as for raising string representations and hence are omitted.

Proposition 6.

(a) $L[n,m]$ *is indecomposable.*
(b) V *is a subrepresentation of* $L[n,m]$ *if and only if* $V \cong L[n-t, m-t]$*, such that* $0 \leqslant t \leqslant n$.
(c) $L[n,m]$ *is irreducible if and only if* $n = 0$.
(d) $L[n,m] \cong L[k,l]$ *if and only if* $n = k$ *and* $m = l$.
(e) The quotient $\mathfrak{e}(3)$*-module* $\frac{L[n,m]}{L[n-t,m-t]}$ *is isomorphic to* $L[t-1, m+n-(t-1)]$ *for* $1 \leqslant t \leqslant n$.

The final proposition of the subsection establishes that raising and lowering strings are distinct families of indecomposable representations (except in the trivial case described below).

Proposition 7. $R[n,m] \cong L[k,l]$ *if and only if* $n = k = 0$ *and* $m = l$.

Proof. If $n = k = 0$ and $m = l$, then it is clear that $R[0,m] \cong L[0,m]$, since in this case the operators P_z, P_+ and P_- act trivially on $R[0,m]$ and $L[0,m]$, and, as $\mathfrak{sl}(2,\mathbb{C})$-modules, $R[0,m] \cong_{\mathfrak{sl}(2,\mathbb{C})} L[0,m] \cong_{\mathfrak{sl}(2,\mathbb{C})} M_m[X,Y]$.

Suppose that $R[n,m] \cong L[k,l]$ with the bijective intertwining operator $\phi : R[n,m] \longrightarrow L[k,l]$. As $\mathfrak{sl}(2,\mathbb{C})$-modules:

$$L[k,l] \cong_{\mathfrak{sl}(2,\mathbb{C})} R[n,m] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{j=0}^{n} M_{n+m-2j}[X,Y]. \tag{31}$$

Thus we have $k = n$ and $l = m$.

The element X^{n+m} of $R[n,m]$ is the unique highest weight vector of weight $n+m$ up to scalar multiple. The element $f_{n,m+n}$ of $L[n,m] = L[k,l]$ is the unique highest weight vector of weight $n+m$ up to scalar multiple. Further, X^{n+m} and $f_{n,m+n}$ are of higher weight (up to scalar multiple) than any other element in their respective module. Hence, there is a nonzero λ such that

$$\phi(X^{n+m}) = \lambda f_{n,m+n}. \tag{32}$$

Thus,

$$0 = \phi(P_- \cdot X^{n+m}) = \lambda P_- \cdot f_{n,m+n}, \tag{33}$$

so that

$$P_- \cdot f_{n,m+n} = 0. \tag{34}$$

However,

$$P_- \cdot f_{n,m+n} \begin{cases} -f_{n-1,m+n-2} & : n \neq 0 \\ 0 & : n = 0. \end{cases} \tag{35}$$

Thus, $n = 0$. In summary, $n = k = 0$ and $m = l$ as required. □

Figure 2 illustrates the action of P_+, P_z and P_- (up to scalar multiple) in $L[2,4] \cong_{\mathfrak{sl}(2,\mathbb{C})} L_2[2,4] \oplus L_1[2,4] \oplus L_0[2,4] \cong_{\mathfrak{sl}(2,\mathbb{C})} M_6[X,Y] \oplus M_4[X,Y] \oplus M_2[X,Y]$. Vertices on the same vertical line form a basis of an $\mathfrak{sl}(2,\mathbb{C})$-irrep. Vertices of the same horizontal height have the same J_z-weight.

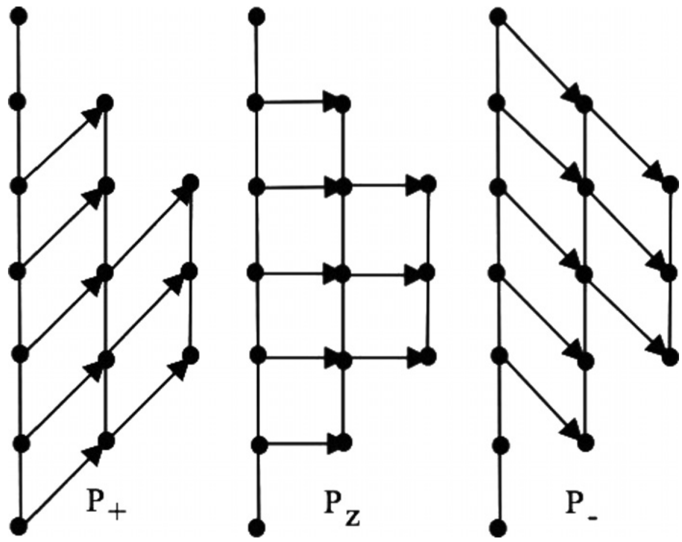

Figure 2. The action of P_+, P_z and P_- (up to scalar multiple) in $L[2, 4]$.

4.3. Parallelogram representations

The representations within the final family of finite-dimensional representations are formed from the tensor product of a raising string representation with a lowering string representation. Specifically, we have the following definition.

Definition 3. *For each $n, m \in \mathbb{Z}_{>0}$, a parallelogram representation, denoted by $P[n, m]$, is the tensor product of $R[n, n]$ with $L[m, m]$. That is, $P[n, m] \equiv R[n, n] \otimes L[m, m]$.*

Remark 2. Let $Hom_{\mathbb{C}}(R[m, m], R[n, n])$ be the space of $\mathbb{C}$-linear maps from $R[m, m]$ to $R[n, n]$. Then, following ([10], p 27), $Hom_{\mathbb{C}}(R[m, m], R[n, n])$ is an $\mathfrak{e}(3)$-module with action $*$ given by

$$X * \phi(v) \equiv X \cdot \phi(v) - \phi(X \cdot v), \tag{36}$$

where $X \in \mathfrak{e}(3)$, $v \in R[m, m]$ and $\phi \in Hom_{\mathbb{C}}(R[m, m], R[n, n])$. We then have an interesting realization of $P[n, m]$:

$$\begin{aligned} P[n, m] &\cong R[n, n] \otimes L[m, m] \cong R[n, n] \otimes R[m, m]^* \\ &\cong R[m, m]^* \otimes R[n, n] \\ &\cong Hom_{\mathbb{C}}(R[m, m], R[n, n]). \end{aligned} \tag{37}$$

That is, as an $\mathfrak{e}(3)$-module,

$$P[n, m] \cong Hom_{\mathbb{C}}(R[m, m], R[n, n]), \tag{38}$$

with the special case $P[n, n] \cong End_{\mathbb{C}}(R[n, n])$.

The following lemmas and remark will be used to prove that $P[n, m]$ is indecomposable in theorem 2 below. Note that when we refer to weight vectors below we mean weight vectors with respect to $\mathfrak{sl}(2, \mathbb{C})$.

The first lemma, lemma 1, is a generalization of theorem 4.2 in [13] (we describe the highest weight vectors for $M_n[X, Y] \otimes M_m[X, Y]$ and [13] describes highest weight vectors in $M_n[X, Y] \otimes M_n[X, Y]$).

Lemma 1. *Let $n \leqslant m$; then the highest weight vector of $M_n[X, Y] \otimes M_m[X, Y]$ of weight $m + n - 2k$, where $0 \leqslant k \leqslant n$, is a nonzero scalar multiple of*

$$w_{m+n-2k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} X^{n-i} Y^i \otimes X^{m-(k-i)} Y^{k-i}. \tag{39}$$

Proof. Any vector w in $M_n[X, Y] \otimes M_m[X, Y]$ of weight $m + n - 2k$, for $k \leqslant n$, has the form

$$w = \sum_{i=0}^{k} a_i X^{n-i} Y^i \otimes X^{m-(k-i)} Y^{k-i}. \tag{40}$$

For w to be a highest weight vector, we must have $J_+ \cdot w = 0$. Therefore,

$$J_+ \cdot w = \sum_{i=0}^{k=0} a_i (J_+ \cdot X^{n-i} Y^i \otimes X^{m-(k-i)} Y^{k-i} + X^{n-i} Y^i \otimes J_+ \cdot X^{m-(k-i)} Y^{k-i}), \tag{41}$$

$$= \sum_{i=1}^{k} a_i (i X^{n-i+1} Y^{i-1} \otimes X^{m-(k-i)} Y^{k-i}) + \sum_{i=0}^{k-1} a_i ((k-i) X^{n-i} Y^i \otimes X^{m-(k-i)+1} Y^{k-i-1}) \tag{42}$$

$$= \sum_{i=1}^{k} a_i (i X^{n-i+1} Y^{i-1} \otimes X^{m-(k-i)} Y^{k-i}) + \sum_{i=1}^{k} a_{i-1} ((k-i+1) X^{n-i+1} Y^{i-1} \otimes X^{m-(k-i)} Y^{k-i}) \tag{43}$$

$$= \sum_{i=1}^{k} (a_i i + a_{i-1}(k-i+1)) X^{n-i+1} Y^{i-1} \otimes X^{m-(k-i)} Y^{k-i} \tag{44}$$

$$= 0. \tag{45}$$

Since the simple tensors are linearly independent, we get the relations

$$a_i i + a_{i-1}(k-i+1) = 0 \tag{46}$$

for $1 \leqslant i \leqslant k$. Hence,

$$a_i = -\frac{k-i+1}{i} a_{i-1}, \quad 1 \leqslant i \leqslant k. \tag{47}$$

Induction on i shows that

$$a_i = (-1)^i \binom{k}{i}, \quad 1 \leqslant i \leqslant k, \tag{48}$$

which completes the proof. □

As an $\mathfrak{sl}(2,\mathbb{C})$-module,

$$P[n,m] \cong_{\mathfrak{sl}(2,\mathbb{C})} \left(\bigoplus_{i=0}^{n} M_{2n-2i}[X,Y]Z^i\right) \otimes \left(\bigoplus_{j=0}^{m} L_{m-j}[m,m]\right) \tag{49}$$

$$\cong_{\mathfrak{sl}(2,\mathbb{C})} \left(\bigoplus_{i=0}^{n} M_{2n-2i}[X,Y]\right) \otimes \left(\bigoplus_{j=0}^{m} M_{2m-2j}[X,Y]\right), \tag{50}$$

so that

$$P[n,m] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{i=0}^{n}\bigoplus_{j=0}^{m} M_{2n-2i}[X,Y] \otimes M_{2m-2j}[X,Y]. \tag{51}$$

Each component in the sum of equation (51) may be decomposed by the Clebsch–Gordan theorem 1:

$$M_{2n-2i}[X,Y] \otimes M_{2m-2j}[X,Y] \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{k=0}^{k'} M_{2(n+m)-2(i+j)-2k}[X,Y], \tag{52}$$

where

$$k' = \begin{cases} 2(n-m)-2(i-j) & : 2n-2i \geqslant 2m-2j \\ 2(m-n)-2(j-i) & : 2n-2i < 2m-2j. \end{cases} \tag{53}$$

Lemma 2. *The highest weight vectors of a fixed weight* λ *in* $P[n,m]$ *have the form*

$$w = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij} w_{ij}, \tag{54}$$

where the a_{ij} *are scalars and* w_{ij} *is the highest weight vector of weight* λ *in* $M_{2n-2i}[X,Y]Z^i \otimes L_{m-j}[m,m] \cong_{\mathfrak{sl}(2,\mathbb{C})} M_{2n-2i}[X,Y] \otimes M_{2m-2j}[X,Y]$ *(if one exists) as described in lemma 1 for* $0 \leqslant i \leqslant n$ *and* $0 \leqslant j \leqslant m$.

Proof. Let w be a highest weight vector of $P[n,m]$ of fixed weight λ. Then w is of the form

$$w = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k} a_{ijk} w_{ijk}, \tag{55}$$

where w_{ijk} is a simple tensor of weight λ from $M_{2n-2i}[X,Y]Z^i \otimes L_{m-j}[m,m]$ for each k and the a_{ijk} are scalars and not all zero. That is, for fixed i and j, the index k of w_{ijk} ranges over the number of weight vectors of weight λ in $M_{2n-2i}[X,Y]Z^i \otimes L_{m-j}[m,m]$. The simple tensors w_{ijk} and $w_{i'j'k'}$ are linearly independent, as are $J_+ \cdot w_{ijk}$ and $J_+ \cdot w_{i'j'k'}$ (if nonzero) if $(i,j) \neq (i',j')$. From this it follows that for fixed i, j,

$$w_{ij} = \sum_{k} a_{ijk} w_{ijk}, \tag{56}$$

is a highest weight vector from $M_{2n-2i}[X,Y]Z^i \otimes L_{m-j}[m,m]$. The result now follows from lemma 1. □

Remark 3. Recall that X^{2n} is a basis element of weight $2n$ in $R[n,n]$ and $f_{m,2m}$ is a basis element of weight $2m$ in $L[m,m]$. Up to scalar multiple, $X^{2n} \otimes f_{m,2m}$ is the unique highest weight vector of weight $2n+2m$ in $P[n,m]$. If $P[n,m]$ were to decompose, then one component of the decomposition must therefore contain $X^{2n} \otimes f_{m,2m}$. That is, if $P[n,m] \cong W_0 \oplus W_1$, then $X^{2n} \otimes f_{m,2m} \in W_0$ or $X^{2n} \otimes f_{m,2m} \in W_1$.

The highest weight vector $X^{2n} \otimes f_{m,2m}$ generates the *terminal* irrep V_T with basis $\{X^{2n-i}Y^i \otimes f_{0,0}\}$, for $0 \leqslant i \leqslant 2n$. Note that V_T is equivalent to the irrep $M_{2n}[X,Y]$. Thus, considering remark 3, we have the following lemma.

Lemma 3. *Let V_T be the irrep with basis $\{X^{2n-i}Y^i \otimes f_{0,0}\}$, for $0 \leqslant i \leqslant 2n$. If $P[n,m] \cong W_0 \oplus W_1$, then $V_T \subseteq W_0$ or $V_T \subseteq W_1$.*

Lemma 4. *Let v be a highest weight vector of $P[n,m]$; then there exists $L \in \mathfrak{e}(3)$ such that $L \cdot v \in V_T - \{0\}$.*

Proof. First we assume that v_{ij} is a highest weight vector of the component $M_{2n-2i}[X,Y]Z^i \otimes L_{m-j}[m,m]$ in the $\mathfrak{sl}(2,\mathbb{C})$-decomposition of $P[n,m]$ from equation (51). Then, by lemma 1, v_{ij} of weight $2n+2m-2i-2j-2k$ for $0 \leqslant k \leqslant min(2m-2j, 2n-2i)$ is given by a nonzero scalar multiple of

$$v_{ij} = \sum_{l=0}^{k} c_{ijl} X^{2n-2i-l} Y^l Z^i \otimes f_{m-j,2m-2j-(k-l)}, \tag{57}$$

where c_{ijl} is a nonzero scalar determined by lemma 1 and equation (36). The exact value of c_{ijl} is not important: for our purposes we only need that it is nonzero. Then,

$$P_z^{i+k} \cdot v_{ij} = P_z^{i+k} \cdot \left(c_{ij0} X^{2n-2i} Z^i \otimes f_{m-j,2m-2j-k}\right), \tag{58}$$

since $P_z^{i+k} \cdot (X^{2n-2i-l} Y^l Z^i \otimes f_{m-j,2m-2j-(k-l)}) = 0$ for $l > 0$. Thus,

$$\begin{aligned} P_-^{m-j-k} P_z^{i+k} \cdot v_{ij} &= (-1)^{i+m-j-k} i! \frac{(j+k)!}{j!} \frac{m!}{(j+k)!} c_{ij0} X^{2n-i} Y^i \otimes f_{0,0}, \\ &= (-1)^{i+m-j-k} i! \frac{m!}{j!} c_{ij0} X^{2n-i} Y^i \otimes f_{0,0} \in V_T - \{0\} \end{aligned} \tag{59}$$

and so $P_-^{m-j-k} P_z^{i+k} \cdot v_{ij} \in V_T - \{0\}$.

We now consider the case of a general highest weight vector v in $P[n,m]$ of weight N, where $0 \leqslant N \leqslant 2m+2n$. Lemma 2 implies that v is in the form

$$v = \sum \lambda_{ij} v_{ij}, \tag{60}$$

where v_{ij} is a highest weight vector of $M_{2n-2i}[X,Y]Z^i \otimes L_{m-j}[m,m]$ in the decomposition of $P[n,m]$ of weight N and not all scalars $\lambda_{ij} = 0$. Note that a highest weight vector v_{ij} of weight N may not exist for all i and j.

For each i, j for which v_{ij} of weight N exists, we have a non-negative integer k_{ij}:

$$N = 2n - 2i + 2m - 2j - 2k_{ij}, \tag{61}$$

where $0 \leqslant k_{ij} \leqslant \min(2m-2j, 2n-2i)$. If there exists a unique maximal $i + k_{ij}$ for which v_{ij} exists and $\lambda_{ij} \neq 0$, then equations (59) and (60) imply

$$P_-^{m-j-k} P_z^{i+k_{ij}} \cdot v = P_-^{m-j-k} P_z^{i+k_{ij}} \cdot (\lambda_{ij} v_{ij}) \in V_T - \{0\}. \tag{62}$$

Otherwise there exists more than one such pair i_s, j_s such that $v_{i_s j_s}$ exists, $\lambda_{i_s,j_s} \neq 0$ and $i + k \equiv i_s + k_{i_s j_s}$ is maximal.

Consider

$$\begin{aligned} P_z^{i+k} \cdot v = \sum_s \lambda_{i_s j_s} (-1)^{i_s} i_s! \frac{(j_s + k_{i_s j_s})!}{j_s!} c_{i_s j_s 0} \\ \times X^{2n-i_s} Y^{i_s} \otimes f_{m-j_s-k_{i_s j_s}, 2m-2j_s-2k_{i_s j_s}}. \end{aligned} \tag{63}$$

Note that if two (or more) simple tensors in equation (63) are linearly dependent (and hence equal) and indexed with s and t, we have $i_s = i_t$, $i_s + k_{i_s,j_s} = i_t + k_{i_t,j_t}$ and $j_s + k_{i_s j_s} = j_t + k_{i_t j_t}$. It follows that $i_s = i_t$, $j_s = j_t$ and $k_{i_s j_s} = k_{i_t j_t}$ so that $v_{i_s j_s} = v_{i_t j_t}$. Hence the terms in the sum of equation (63) are distinct and $P_z^{i+k} \cdot v \neq 0$.

Let $m - j' - k' \equiv m - j_{s'} - k_{i_{s'} j_{s'}}$ be maximal among those terms in equation (63) and consider

$$P_-^{m-j'-k'} P_z^{i+k} \cdot v = \sum_{s'} \lambda_{i_{s'} j_{s'}} (-1)^{m-j_{s'}-k_{i_{s'},j_{s'}}+i_{s'}} (i_{s'}!) \frac{m!}{j_{s'}!} c_{i_{s'} j_{s'} 0} \times X^{2n-i_{s'}} Y^{i_{s'}} \otimes f_{0,0}. \tag{64}$$

The element in equation (64) is in V_T, and we must show that it is nonzero. Note that if two (or more) simple tensors in equation (64) are linearly dependent (and hence equal) and indexed with s' and t', we have $i_{s'} = i_{t'}$, $i_{s'} + k_{i_{s'} j_{s'}} = i_{t'} + k_{i_{t'} j_{t'}}$ and $i_{s'} + j_{s'} + k_{i'_s j'_s} = i_{s'} + j_{t'} + k_{i_{t'} j_{t'}}$, with the last equality a consequence of equation (61). It follows that $i'_s = i'_t$, $j'_s = j'_t$, and $k_{i'_s j'_s} = k_{i'_t j'_t}$ so that $v_{i'_s j'_s} = v_{i'_t j'_t}$. Thus all simple tensors in equation (64) are linearly independent and in V_T.

Hence,

$$P_-^{m-j'-k'} P_z^{i+k} \cdot v \in V_T - \{0\}, \tag{65}$$

which completes the proof. □

Theorem 2. *$P[n, m]$ is indecomposable.*

Proof. By way of contradiction, suppose that $P[n, m]$ decomposes with decomposition $P[n, m] \cong W_0 \oplus W_1$ such that $W_0 \neq 0$ and $W_1 \neq 0$. Then, $V_T \subseteq W_0$ or $V_T \subseteq W_1$ by lemma 3. Without loss of generality, let $V_T \subseteq W_1$. Since $W_0 \neq 0$ and is an $\mathfrak{sl}(2, \mathbb{C})$-module (since it is an $\mathfrak{e}(3)$-module), it must contain a highest weight vector v. By lemma 4, there exists $L \in \mathfrak{e}(3)$ such that $L \cdot v \subseteq V_T - \{0\}$; hence $L \cdot v \in W_1 - \{0\}$. Since $L \cdot v \in W_0 - \{0\}$, we have $L \cdot v \in W_0 \cap W_1$ with $L \cdot v \neq 0$, a contradiction. Thus, it must be the case that $P[n, m]$ is indecomposable. □

The next proposition establishes that in creating the family of parallelogram representations we have indeed created a family of representations distinct from the raising and lowering strings.

Proposition 8. *$P[n, m]$ is neither equivalent to a raising nor a lowering string representation.*

Proof. The $P[n, m]$ element $Z^n \otimes f_{0,0}$ generates the subrepresentation $R[n, n]$, while the element $X^{2n} \otimes f_{m,2m}$ generates the subrepresentation $L[m, 2n + m]$. Propositions 2 and 6 thus imply that $P[n, m]$ is neither a raising string nor a lowering string representation. □

A simple consideration of dimensions establishes the following proposition.

Proposition 9.

(a) $P[n, n] \cong P[m, m]$ if and only if $n = m$,
(b) $P[n, m] \cong P[n, k]$ if and only if $m = k$,
(c) $P[n, m] \cong P[k, m]$ if and only if $n = k$.

Proposition 9 implies that the family of parallelogram representations is infinite. Finally, we show that the family of parallelogram representations is closed under the dual operation.

Proposition 10. $P[n, m]^* \cong P[m, n]$.

Proof. $P[n, m]^* \cong (R[n, n] \otimes L[m, m])^* \cong R[n, n]^* \otimes L[m, m]^* \cong L[n, n] \otimes R[m, m] \cong R[m, m] \otimes L[n, n] \cong P[m, n]$. □

5. Concluding remarks

In a subsequent paper, we will construct finite-dimensional, indecomposable representations of $\mathfrak{e}(m)$ for $m > 3$. We will consider cases $m = 4$, $m = 6$, $m = 2n$ for $n \geqslant 4$, and $m = 2n+1$ for $n \geqslant 2$ separately. The cases correspond to $\mathfrak{so}(m)_{\mathbb{C}}$ being a Lie algebra of type $A_1 \times A_1$, A_3, D_n or B_n, respectively. In each case, the indecomposable, finite-dimensional representations of $\mathfrak{e}(m)$ will be constructed from finite-dimensional, irreducible representations of $\mathfrak{so}(m)_{\mathbb{C}}$.

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